

1. **Two flavors neutrino oscillation in vacuum:**a) A neutrino ν_α ($\alpha=e$ or μ) state at a time t is

$$|\nu_\alpha(t)\rangle = \sum_{i=1,2} U_{\alpha i}^* e^{-\frac{i}{\hbar}(E_i t - \vec{p}_i \cdot \vec{x})} |\nu_i\rangle$$

where $U_{\alpha i}^*$ are the matrix elements of a unitary matrix

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Assume at a time $t = 0$ the flavor eigenstate $|\nu_\alpha\rangle$ was produced. Derive the probability formulas:

$$P(\nu_e \rightarrow \nu_\mu; t = L/c) = |\langle \nu_\mu(0) | \nu_e(L/c) \rangle|^2 = \sin^2(2\theta) \sin^2 \left[\frac{1.267 \Delta m^2 [\text{eV}^2] L [\text{km}]}{E_\nu [\text{GeV}]} \right]$$

$$P(\nu_e \rightarrow \nu_e; t = L/c) = |\langle \nu_e(0) | \nu_e(L/c) \rangle|^2 = 1 - \sin^2(2\theta) \sin^2 \left[\frac{1.267 \Delta m^2 [\text{eV}^2] L [\text{km}]}{E_\nu [\text{GeV}]} \right]$$

with $\Delta m^2 = m_2^2 - m_1^2$.Hint: In the ultrarelativistic limit ($m_i c \ll p_i$)

$$E_i \simeq p_i c + \frac{m_i^2 c^4}{2E} \text{ and } \vec{p}_i \cdot \vec{x} \simeq p_i c t.$$

b) Draw $P(\nu_e \rightarrow \nu_\mu)$ in vacuum, when $E_\nu = 1$ MeV and $L = 0$ to 2 km. Use the values $\sin(2\theta_{13}) = 0.092$ and $|\Delta m_{31}^2| \approx |\Delta m_{32}^2| \equiv \Delta m_{atm}^2 = 2.43 \times 10^{-3} \text{ eV}^2$.

How the neutrino energy and the distance affect the oscillation?

2. **Supernova Neutrinos I:**

(a) Using the relations

$$v = \frac{pc}{E} \text{ and } E^2 = p^2 c^2 + m^2 c^4$$

for the velocity v and the energy E of a relativistic neutrino, where p and m are the momentum and the rest mass of the neutrino, respectively, derive the equation

$$\Delta t [\text{s}] = 0.0515 \left(\frac{mc^2 [\text{eV}]}{E [\text{MeV}]} \right)^2 \times L [\text{kpc}]$$

for the time-of-flight delay of massive neutrinos (from supernova explosions).
 Hint. $mc^2 \ll E$ and $(1-x)^{1/2} = 1 - \frac{1}{2}x$ (x small).

(b) Estimate, using the above-derived equation, the upper limit for the neutrino mass from the neutrino bursts observed by Kamiokande II and IMB detectors [lecture notes slide 10.37]. The distance to the SN1987A is 51.8 kpc.

3. Supernova Neutrinos II:

Supernova neutrino spectrum can be expressed as

$$f_\nu(E_\nu) = \frac{1}{T_\nu^3 F_2(0)} \frac{E_\nu^2}{\exp(E_\nu/T_\nu) + 1}$$

with $T_\nu = 4, 5$ and 8 MeV for $\nu_e, \bar{\nu}_e$, and ν_x ($x = \mu, \tau$), respectively.

Assume a spherical detector of the radius of $r = 3$ metres and filled with xenon gas at the pressure of $P = 10$ bar. The interaction process being the coherent neutrino nucleus scattering in the xenon gas, the cross section can be expressed as

$$\sigma = C \times N_n^2 \times E_\nu^2,$$

with xenon obeying the ideal-gas law ($C = 4.242 \times 10^{45}$ cm²/MeV², $N_n=76$).

The total neutrino energy release can be taken as $W_\nu = 2 \times 10^{59}$ MeV, equally partitioned to all six flavours.

How many neutrino events such a detector would observe from standard supernova? No oscillations assumed.

Hints. See lecture notes [slides 10.16 and 10.17]. Luminosity at the Earth is

$$L_\nu = \frac{1}{4\pi D^2} \frac{W_\nu}{\langle E_\nu \rangle},$$

where $\langle E_\nu \rangle = T_\nu F_3(0)/F_2(0)$ is the mean neutrino energy. Fermi integrals are

$$F_2(0) = \int \frac{x^2 dx}{\exp(x) + 1} \approx 1.80300, \quad F_3(0) = \int \frac{x^3 dx}{\exp(x) + 1} \approx 5.68219,$$

and

$$F_4(0) = \int \frac{x^4 dx}{\exp(x) + 1} \approx 23.33025.$$