Task: Describe plasma theory in 2 hours

Impossible? No!

Compare with general relativity

\[ G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \text{and that’s it!} \]

But if you want to describe the dynamics of two colliding rotating black holes, well, that is another matter.

One can claim that all plasma physics is described by Boltzmann’s equation

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{q}{m} (E + v \times B) \cdot \frac{\partial f}{\partial v} = \left( \frac{\partial f}{\partial t} \right)_e \]

With this only you do not get far to solve any relevant plasma physics problems.
What does Boltzmann’s equation tell about plasma physics?

\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial f}{\partial \mathbf{v}} \cdot e \]

- **distribution function:** plasma physics is statistical physics
- **role of EM forces:** plasma physics is electrodynamics
- **collisions can break the conservation of phase space density**

How to define the plasma state?

**Plasma is quasi-neutral ionized gas containing enough free charges to make collective electromagnetic effects important for its physical behaviour.**

The most fundamental plasma properties are:
- Debye screening
- Plasma oscillations
- Gyro motion of plasma particles

**Debye screening**

\[ \varphi = \frac{q}{4\pi\varepsilon_0 r} \exp \left( -\frac{r}{\lambda_D} \right) \]

Assume thermal equilibrium (Boltzmann distribution)

\[ n_\alpha(r) = n_\alpha \exp \left( \frac{q_e \varphi}{k_B T_\alpha} \right) \]

- \( \alpha \) labels the particle populations (e.g., \( e, p \))

Introduce a test charge \( q_T \). What will be its potential?

Home exercise:

\[ \varphi = \frac{q_T}{4\pi\varepsilon_0 r} \exp \left( -\frac{r}{\lambda_D} \right) \]

- \( \lambda_D^2 = \frac{1}{\epsilon_0} \sum \frac{n_\alpha q^2_\alpha}{k_B T_\alpha} \)

**Debye length**

\[ \lambda_D \propto \sqrt{\frac{T}{n}} \]

Number of particles in a Debye sphere:

\[ N = \frac{4\pi}{3} n_0 \lambda_D^3 \]

A little better (?) definition for plasma

\[ \frac{1}{\sqrt{n_0}} \ll \lambda_D \ll L \]

\( L \) is the size of the system

**Plasma parameter**

\[ \Lambda = n_0 \lambda_D^3 \gg 1 \]
**Plasma oscillation**

Assume: $n_0$ fixed ions (+) & $n_0$ moving electrons (–)

Apply a small electric field $E_1$

$\rightarrow$ electrons move:

$$n_+ = n_0$$
$$n_- = n_0 + n_1(r,t) ; \quad n_1 \ll n_0$$

Electron continuity equation:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) = 0$$

$$n = n_0 + n_1(r,t) ; \quad \mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1(r,t)$$

$u_0 = 0 \leftrightarrow$ electrons are assumed cold

$$\frac{\partial n_0}{\partial t} + \nabla \cdot ((n_0 + n_1) \mathbf{u}_1) = 0 \Rightarrow \frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \mathbf{u}_1 + \nabla \cdot (n_1 \mathbf{u}_1) = 0$$

Linearized continuity equation (1st order terms only): $\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \mathbf{u}_1 = 0$ !!

Force: $F = qE \Rightarrow m \frac{\partial \mathbf{u}_1}{\partial t} = -eE_1$

1st Maxwell: $\nabla \cdot \mathbf{E}_1 = -c n_1/\epsilon_0$

$$\Rightarrow \frac{\partial^2 n_1}{\partial t^2} + \left( \frac{m c^2}{\epsilon_0 m_e} \right) n_1 = 0$$

$$\omega_{pe}^2 = \frac{n e^2}{\epsilon_0 m_e} \text{ plasma frequency}$$

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**Useful rules-of-thumb**

Plasma frequency (angular frequency)

$$\omega_{pe}^2 = \frac{n e^2}{\epsilon_0 m_e}$$

$$f_{pe} (Hz) = \frac{\omega_{pe}}{2\pi} \approx 9.0 \cdot \sqrt{n/(m^{-3})}$$

Debye length

$$\lambda_D (m) \approx 7.4 \sqrt{T(eV)/(n/(cm^{-3}))}$$

Note the units!

$1 \text{ eV} \approx 1.16 \cdot 10^4 \text{ K}$

Gyromotion in the magnetic field

$$F = q_0 (v \times B) \Rightarrow$$

$$\omega_{co} = \frac{q_0 B}{m_0} ; \quad f_{co} = \frac{\omega_{co}}{2\pi}$$

$$f_{pe} (Hz) \approx 28 \cdot B(nT)$$

$$f_{pi} (Hz) \approx 1.5 \cdot 10^{-2} \cdot B(nT)$$
Plasma physics is difficult – but why?

- Combination of statistical physics and electromagnetism
- Large variety of scales, from electrons to ions to fluids
- A great variety of plasma descriptions must be mastered
  - single particle motion
  - Vlasov theory (electrons and ions described by distribution functions)
  - fluid descriptions (e.g., magnetohydrodynamics)
  - various hybrids of these
- Collisions or their absence

Electrodynamics: Maxwell’s equations

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \rho / \varepsilon_0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\n\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\n\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

Magnetic flux \( \Phi = \int_S \mathbf{B} \cdot d\mathbf{a} \) is important in macroscopic plasma physics

EM fields are empirically determined through the Lorentz force

\[
\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
\text{or} \\
\mathbf{F} = \frac{d}{dt}(\gamma m \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

Because \( \mathbf{v} \cdot \mathbf{F} = q(\mathbf{v} \cdot \mathbf{E} + \mathbf{v} \cdot \mathbf{v} \times \mathbf{B}) = q(\mathbf{v} \cdot \mathbf{E}) \)

only \( \mathbf{E} \) performs work on charges \( \frac{dW}{dt} = \frac{d}{dt}(\gamma m c^2) = q\mathbf{E} \cdot \mathbf{v} \)

Thus any “magnetic acceleration” is associated with an electric field in the frame of reference where the acceleration is observed
Ohm’s law

Ohm’s law \( \mathbf{J} = \sigma \mathbf{E} \) relating the electric current and electric field is similar to the other constitutive equations \( \mathbf{D} = \varepsilon \mathbf{E} \) and \( \mathbf{B} = \mu \mathbf{H} \).

The conductivity \( \sigma \), permittivity \( \varepsilon \), and permeability \( \mu \) depend on the electric and magnetic properties of the media considered. They may be scalars or tensors, and there does not need to be a local constitutive relation at all, not even Ohm’s law!

A medium is called linear if \( \varepsilon, \mu, \sigma \) are scalars and they are not functions of time and space.

Note that also in linear media \( \mathbf{H} = \mathbf{H} (Z, \omega) \), which is a very important relationship in plasma physics!

Conservation of EM energy

Poynting’s theorem

The energy of electromagnetic field is given by

\[ W_{EM} = \frac{1}{2} \int (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \, d^3 r \]

\[ w_E = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} \quad w_M = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \]

Strating from Maxwell’s equations it is straightforward to get

\[ \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \]

Poynting’s theorem

where \( \mathbf{S} = \mathbf{E} \times \mathbf{H} \) is the Poynting vector.

Integrating over volume \( V \) (and using Gauss’s law for the divergence)

\[ - \int_V \mathbf{J} \cdot \mathbf{E} \, d^3 r = \int_{\partial V} \mathbf{S} \cdot d\mathbf{a} + \int_V \frac{\partial}{\partial t} (w_E + w_M) \, d^3 r \]

work performed by the EM field  energy flux through the surface of \( V \)  change of energy in \( V \)
Example: Poynting’s theorem in fluid plasma (MHD)

\[- \oint_{\partial V} \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} = \frac{\partial}{\partial t} \int_{V} \mathbf{B}^2 / 2 \, d^3r + \int_{V} \frac{\mathbf{J}^2}{\sigma} \, d^3r + \int_{V} \mathbf{V} \times \mathbf{B} \, d^3r\]

- EM energy flux into (out of) volume \( V \)
- change of magnetic energy in volume \( V \)
- plasma heating in volume \( V \)
- acceleration in volume \( V \)

Single-particle motion: Guiding centre approximation

Equation of motion of charged particles is

\[\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{F}_{\text{non-EM}}\]  
(assume, for the time being, nonrelativistic motion; \( \gamma = 1 \) and \( \mathbf{p} = mv \))

Consider the case \( \mathbf{E} = 0 \) and \( \mathbf{B} = \text{const} \) (neglect the non-EM forces)

The radius of the circle is \( r_L = \frac{v_\perp}{|\omega_c|} = \frac{mv_\perp}{|q|B} \)  
\( \omega_c = \frac{qB}{m} \)

The gyro period (cyclotron period, Larmor time) is: \( \tau_L = \frac{2\pi}{|\omega_c|} \)

The pitch angle \( (\alpha) \) of the helical path is defined by

\[\tan \alpha = \frac{v_\perp}{v_\parallel} \]

\[\alpha = \arcsin(v_\perp/v) = \arccos(v_\parallel/v)\]
The frame of reference where $v_{||} = 0$ : Guiding centre system (GCS)

Decomposition of the motion to the motion of the guiding centre and to the gyro motion is called the guiding centre approximation

In the GCS the charge causes an electric current: $I = q / r_L$

The magnetic moment associated to the circular loop is

$$\mu = I \times r_L = \frac{1}{2} \frac{q^2 r_L^2 B}{m} = \frac{1}{2} \frac{mv_{\perp}^2}{B} = \frac{W_{\perp}}{B}$$

or, in the vector form

$$\mu = \frac{1}{2} q r_L \times v_{\perp}$$

Clearly: $\mu$ is always opposite to $B$ ($r_L$ depends on the sign of $q$)

Thus plasma can be considered a diamagnetic medium:

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**E x B drift**

Let $E = \text{const}$ and $B = \text{const}$

The eq. of motion along $B$ is $m \ddot{v}_{||} = q\dot{E}_{||}$

$\rightarrow$ constant acceleration parallel/antiparallel to $B$

$\rightarrow$ very rapid cancellation of large-scale $E_{||}$ in plasma!

The perpendicular components of the eq. of motion are

$$\dot{v}_x = \omega_c v_y = \frac{q}{m} E_x$$

$$\dot{v}_y = -\omega_c v_x$$

Substitution $v_{y'} = v_y + E_x / B$ leads again to gyro motion but now the GC drifts in the $y$-direction with speed $E_x / B$

In vector form:

$$v_{E'} = \frac{E \times B}{B^2}$$

All charged particles drift to the same direction $\perp E$ and $\perp B$
Other non-magnetic drifts

Write the perpendicular eq. of motion in the form \( \frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m} (\mathbf{v}_\perp \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m} \)
Assume that \( \mathbf{F}_\perp \) gives rise to a drift \( \mathbf{v}_s \), and transform \( \mathbf{v}_\perp = \mathbf{v}_s + \mathbf{v}_D \)

\( \Rightarrow \frac{d\mathbf{v}_s}{dt} = \frac{q}{m} (\mathbf{v}_s \times \mathbf{B}) + \frac{q}{m} (\mathbf{v}_D \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m} \)

In GCS the last two terms must sum to 0 \( \Rightarrow \mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2} \) (*)

This requires \( F/\rho B \ll c \). If \( F > \rho q B \), the GC approximation cannot be used!

Inserting \( \mathbf{F}_\perp = q \mathbf{E} \) into (*) we get the \( \mathbf{E} \times \mathbf{B} \)-drift

\( \mathbf{v}_p = \frac{mg \times \mathbf{B}}{qB^2} \times \frac{m}{q} \) separates charges \( \rightarrow \) current

Slow time variations in \( \mathbf{E} \rightarrow \) polarization drift

The corresponding polarization current is

\( \mathbf{J}_p = n_e (\mathbf{v}_p - \mathbf{v}_e) = \frac{n_e (m_e + m_i) \mathbf{E}_\perp}{B^2} \frac{d\mathbf{E}_\perp}{dt} + \frac{n_e m_i \mathbf{E}_\perp}{B^2} \frac{d\mathbf{E}}{dt} \) carried by ions!

Magnetic drifts

Assume static but inhomogeneous magnetic field. Guiding centre approximation is useful if the spatial and temporal gradients of \( \mathbf{B} \) are small as compared to the gyro motion:

\( |\partial \mathbf{B}/\partial t| \ll \omega_e B \ ; \ |\nabla B|_z \ll B/r_L \ ; \ |\nabla B|_\parallel \ll (\omega_e/\nu_\parallel)B \)

Considering first the gradient of \( \mathbf{B} \) only the force on the guiding centre can be shown to be \( \mathbf{F} = -\mu \nabla \mathbf{B} \)

The parallel force gives acceleration along \( \mathbf{B} \)

\( \frac{d\mathbf{v}_D}{dt} = \frac{\mu}{m} \nabla \mathbf{B} \)

From the equation

\( \mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2} \)

we get the gradient drift velocity

\( \mathbf{v}_G = \frac{\mu}{qB^2} \mathbf{B} \times (\nabla \mathbf{B}) = \frac{W_\perp}{qB^2} \mathbf{B} \times (\nabla \mathbf{B}) \)
The field-line curvature (centrifugal force) leads to curvature drift

\[ \mathbf{v}_C = \frac{-m \omega}{qB^2} \mathbf{R}_C \times \mathbf{B} = \frac{m \omega^2}{qB^2} \mathbf{B} \times (\mathbf{B} \cdot \nabla) \mathbf{B} \]

curvature radius

If there are local currents in plasma, i.e., \( \nabla \times \mathbf{B} = 0 \)
the curvature drift reduces to

\[ \mathbf{v}_C = \frac{2W}{qB^3} \mathbf{B} \times \nabla \mathbf{B} \]

and we can combine the gradient and curvature drifts

\[ \mathbf{v}_{GC} = \frac{W}{qB^3} \mathbf{B} \times \nabla \mathbf{B} = W \left( \frac{1}{qBR_C} (1 + \cos^2 \alpha) \mathbf{n} \times \mathbf{t} \right) \]

These are straightforward to modify for relativistic motion by substitution \( \gamma m \rightarrow m \)

Adiabatic invariants

Symmetry principles: periodic motion \( \leftrightarrow \) conserved quantity
symmetry \( \leftrightarrow \) conservation law

What if the motion is almost periodic?

Hamiltonian mechanics:

Let \( q \) & \( p \) be canonical variables and the motion almost periodical \( \Rightarrow \)

\[ I = \oint p dq \quad \text{is constant, called adiabatic invariant} \]

Example: Consider a charged particle in Larmor motion.
Assume that the \( \mathbf{B} \) does not change much during within one circle.
The canonical coordinate is \( r_L \) and the canonical momentum \( p = mv + qA \)

\[
I = \oint p_L \cdot d\mathbf{r}_L = \oint mv_L \cdot d\mathbf{r}_L + q \oint (\nabla \times A) \cdot d\mathbf{S}
\]

\[ = \int_0^{2\pi} mv_L \cdot dl + q \int S \mathbf{B} \cdot d\mathbf{S} \]

\[ = 2\pi mv_L r_L - \frac{|q|B \pi^2 L^2}{|A|} \quad \text{The magnetic moment is an adiabatic invariant} \]
Magnetic mirror

In guiding centre approximation both $W$ and $μ$ are conserved. If $B$ increases slowly, $W'_i$ increases slowly, thus $W_i$ decreases.

What happens when $W_i \to 0$?

\[
μ = \frac{m v^2 \sin^2 α}{2B}
\]

As $μ$ and $v^2$ are conserved, $α$ and $B$ are related through

\[
\frac{\sin^2 α_1}{\sin^2 α_2} = \frac{B_1}{B_2}
\]

When $α \to \pi/2$, the force $F = -μ\nabla ||B$

on the GC turns the charge back (mirror force) and the mirror field $B_m$ for a charge that at $B_0$ has the pitch-angle $α$ is given by

\[
\sin^2 α = B_0/B_m
\]

Also parallel electric field and/or gravitation may need to be considered.

If the non-magnetic forces can be derived from a potential $U(s)$,

\[
\frac{dv_i}{dt} = qE_i + mg_0 - μ\nabla ||B
\]

\[
\frac{dv_i}{dt} = -\frac{∂}{∂s}[U(s) + μB(s)]
\]

Magnetic bottle

A simple magnetic bottle consists of two mirrors facing each other. A charged particle is trapped in the bottle if

\[
\arcsin \sqrt{\frac{B_0}{B_m}} \leq α_0 \leq 180 - \arcsin \sqrt{\frac{B_0}{B_m}}
\]

Otherwise it is said to be in the loss-cone and escape at the end of the bottle.

The dipole field of the Earth is a large magnetic bottle.

Note that there are much more complicated trapping schemes (e.g. tokamak)
In plasma physics $\mu$ is called the first adiabatic invariant.

Consider the bounce period of a charge in a magnetic bottle
\[
\tau_b = 2 \int_{s_m}^{s_m} \frac{ds}{v(s)} = 2 \int_{s_m}^{s_m} \frac{ds}{v(s) (1 - B(s)/B_m)^{1/2}}
\]
If $\tau_b \frac{dB}{dt} / B \ll 1$ then $J = \oint_{\gamma(s)} ds$ is constant (second adiabatic invariant)
This, of course requires that $\tau_b \gg \tau_L$

If the perpendicular drift of the GC is nearly periodic (e.g., in a dipole field), the magnetic flux through the GC orbit
\[ \Phi = \oint A \cdot ds \] is conserved. This is the third adiabatic invariant.

The adiabatic invariants can be used as coordinates in studies of the evolution of the distribution function
\[ f = f(\mu, J, \Phi) \]

### Betatron acceleration

Let $T$ be the kinetic energy of the particle in a time-dependent $B$

Write the time derivative in a moving frame as
\[ \frac{d}{dt} = \partial_t + \mathbf{w} \cdot \nabla \]

In GCS:
\[ \frac{dT_{GCS}}{dt} = \mu \frac{dB}{dt} = \mu \left( \frac{\partial B}{\partial t} + \mathbf{w} \cdot \nabla B + \frac{\partial B}{\partial s} \right) \]

In the reference frame of the observer (OFR)
\[ \frac{dT_{OFR}}{dt} = \frac{dT_{GCS}}{dt} + \frac{1}{2} \frac{d}{dt} \left( \frac{1}{2} \mathbf{w} \cdot \mathbf{E} \right) + \frac{d}{dt} \left( \frac{1}{2} \frac{1}{\mu} \frac{\partial B}{\partial s} \right) \]

Do a little algebra $\Rightarrow$ betatron acceleration:
\[ \frac{dT_{OFR}}{dt} = \mu \frac{\partial B}{\partial t} + q \mathbf{w} \cdot \mathbf{E} \]

Two effects:
- Field-aligned acceleration, if $E_1 \neq 0$.
- Drift betatron:
  Particle drifts toward increasing $B$:
  \[ B_2 > B_1 \]
  Conservation of $\mu$ $\Rightarrow$
  \[ \frac{W_{22}}{W_{11}} = \frac{B_2}{B_1} \]
  thus
  \[ W_{22} > W_{11} \]
Fermi acceleration of cosmic rays

Fermi proposed the drift betatron acceleration as a mechanism to accelerate cosmic rays to very large energies in 1949 in the following form:

Let the particle move in a mirror field configuration where the mirror points move toward each other. Assume that $J$ is conserved.

Now $\frac{d}{ds}$ decreases. To compensate this $v_\parallel$ must increase. Compare this to the acceleration of a tennis ball hit by a racket!

The modern version of Fermi acceleration is called diffusive shock acceleration where shock waves are responsible for the acceleration. It does not conserve $\mu$ and $J$.

Cosmic Rays

<table>
<thead>
<tr>
<th>Type</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>galactic</td>
<td>$&gt; 100$ MeV</td>
</tr>
<tr>
<td>solar</td>
<td>$&lt; 1$ GeV</td>
</tr>
<tr>
<td>Anomalous</td>
<td>around 10 MeV</td>
</tr>
</tbody>
</table>

Spectrum of galactic cosmic rays
A plasma particle \((i)\) is at time \(t\) in location \(r_i(t)\) and has velocity \(v_i(t)\).

The distribution function \(f(r, v, t)\) gives the particle number density in the \((r, v)\) phase space element \(dxdydzdv\) at time \(t\).

The units of \(f\): volume\(^{-1}\) \times (volume of velocity space)\(^{-1}\) = \(s\)\(^3\)\(m\)\(^{-6}\).

Normalization: \(\int_0^\infty \int_V f(r, v, t) r^2 v^2 dv \, dr = N \leftarrow \text{total number of particles}\)

Average density: \(\langle n \rangle = N/V\); density at location \(r\): \(n(r, t) = \int f(r, v, t) r^2 v^2 dv\).

Example: Maxwellian distribution

\[
f(v) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m|v - v_0|^2}{2k_B T} \right)
\]

Normalization: total number of particles 

\(\int_0^\infty \int_V f(r, v, t) r^2 v^2 dv \, dr = N \leftarrow \text{total number of particles}\)
Magnetic field-aligned beam (e.g., particles causing the aurora):

\[ f(v_{\perp}, v_{\parallel}) = n \left( \frac{m}{2\pi k_B T_{\perp}} \right)^{3/2} \exp \left( -\frac{m v_{\perp}^2}{2k_B T_{\perp}} - \frac{m (v_{\parallel} - v_0)^2}{2k_B T_{\parallel}} \right) \]

Loss-cone distribution in a magnetic bottle:

Kappa distribution \( \sim \) Maxwellian with high-energy tail

The tail follows a power law:

\[ f_s(W) = n \left( \frac{m}{2\pi \gamma V_0} \right)^{3/2} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - 1/2)} \left( 1 + \frac{W}{n V_0} \right)^{-(\kappa+1)} \]

\[ [f_s] = m^{-6} \gamma^3 \]

flux

energy

Vlasov equation (VE)

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial v} + q \left( E + v \times B \right) \cdot \frac{\partial f}{\partial v} = 0 \]

Compare with the Boltzmann equation in statistical physics (BE)

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial v} + F \cdot \frac{\partial f}{\partial v} = \left( \frac{\partial f}{\partial v} \right)_c \]

Boltzmann derived \( (\partial f/\partial t)_c \) for strong short-range collisions

In plasmas most collisions are long-range small-angle collisions. They are taken care by the average Lorentz force term

\[ \left( \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial v} + q \left( E + v \times B \right) \cdot \frac{\partial f}{\partial v} - \left( \frac{\partial f}{\partial v} \right)_c \right) \]

VE is often called collisionless Boltzmann equation

(M. Rosenbluth: actually a Boltzmann-less collision equation!)
Vlasov theory

How to solve

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m} (E + v \times B) \cdot \frac{\partial f}{\partial v} = 0
\]

Very hard task in a general case. VE is nonlinear, thus we linearize:

Consider a homogeneous, field-free plasma \( E_0 = B_0 = 0 \)
in an electrostatic approximation:

\[
E_1 = -\nabla \varphi_1; \quad B_1 = 0
\]

The linearized VE is now

\[
\frac{\partial f_{a1}}{\partial t} + v \cdot \frac{\partial f_{a1}}{\partial \mathbf{r}} - \frac{q_a}{m_a} \frac{\partial \varphi_1}{\partial \mathbf{r}} \cdot \frac{\partial f_{a0}}{\partial \mathbf{v}} = 0
\]

where

\[
\nabla^2 \varphi_1 = \frac{1}{\epsilon_0} \sum \frac{n_{a0} k_0}{a} \int f_{a1} d^3v
\]

Vlasov tried this at the end of the 1930s using Fourier transformation

\[
\int_{-\infty}^{\infty} \frac{\partial f_{a0}}{\partial \omega} d\omega - \frac{k_v}{\omega - k_v}
\]

pole along the integration path, what to do?
In 1945 Vlasov presented a solution of VE at the long wavelength limit \( \omega \gg k v \):
\[
\omega^2 \approx \omega_p^2 + \frac{3}{2} k^2 v_n^2 \quad \text{Langmuir wave}
\]

Thus in finite temperature the plasma oscillation propagates as a wave.

In 1946 Lev Landau found the way to handle the pole at \( \nu = \omega/k \).
He used the Fourier method in space but treated the problem as an initial value problem and used Laplace transform in time.

The solution for \( E \) is:
\[
E = E_0 \exp[-i(\omega + i\gamma)t] \propto \exp(\gamma t)
\]

For Maxwellian \( f_0 \gamma < 0 \) and the wave is damped: Landau damping

For an interested student: the long wavelength solution is
\[
\omega_0 \approx \omega_{pe}(1 + 3k^2 \lambda_D^2) \approx \omega_{pe}(1 + \frac{3}{2} k^2 \lambda_D^2)\]

For details, see any advanced plasma physics text book

More realistic configurations soon become manually intractable, already for uniformly magnetized plasma the problem is to solve

\[ |\mathcal{K}(\omega, k)| = 0 \]

where

\[
\mathcal{K}(\omega, k) = \left( 1 - \sum_{\alpha} \frac{\omega_p^2}{\omega^2} \right) I - \sum_{\alpha} \sum_{n=-\infty}^{\infty} \frac{2m_\alpha^2}{n_i^2 \omega^2} 
\int \int_{-\infty}^{\infty} v_\perp dv_\perp \left( k_v - \frac{n_i \omega_0 \partial f_{00}}{v_\perp \partial v_\perp} \right) S_{min}(v_i, v_\perp)
\]

\[
S_{min}(v_i, v_\perp) = \begin{bmatrix}
\frac{n_i^2 \omega_0^2}{k_i^4} J_n^2 & i n_i v_i |\omega_{ce}| J_n J_n' & i n_i |\omega_{ce}| J_n J_n' \\
-\frac{n_i^2 |\omega_{ce}|}{k_i^4} J_n J_n' & v_i^2 J_n^2 & -i n_i v_i |\omega_{ce}| J_n J_n' \\
\end{bmatrix}
\]

\[
J_n' = \frac{dJ_n}{d(k_v v_\perp / |\omega_{ce}|)}
\]
Perpendicular modes on dispersion surfaces

"electron modes"
- O-mode
- X-modes
- upper hybrid mode
- electron Bernstein modes

"ion modes"
- ion Bernstein modes
- lower hybrid mode
- electrostatic ion cyclotron modes (nearly perp.)
- fast MHD mode (magnetosonic)

Parallel modes on dispersion surfaces

"electron modes"
- R-mode
- L-mode
- Langmuir wave
- EM electron cyclotron wave
- whistler mode

"ion modes"
- ion-acoustic wave
- whistler mode
- EM ion cyclotron wave
- Alfven wave
Macroscopic theory: Velocity moments of \( f \)

\[
\int f \, d^3v ; \quad \int v f \, d^3v ; \quad \int vv f \, d^3v
\]

\( n(r, t) = \int f(r, v, t) \, d^3v \)
Density is the zeroth moment; \([n] = \text{m}^{-3}\)

The first moment (\( \alpha \) denotes different particle species):

\[
\Gamma_\alpha(r, t) = \int v f_\alpha(r, v, t) \, d^3v \quad \text{Particle flux; } \{\Gamma\} = \text{m}^2 \text{s}^{-1}
\]

\[
\mathbf{V}_\alpha(r, t) = \frac{\int v f_\alpha(r, v, t) \, d^3v}{\int f_\alpha(r, v, t) \, d^3v} \quad \text{Average velocity = flux/density, } \{\mathbf{V}\} = \text{m} \text{s}^{-1}
\]

**DO NOT EVER MIX UP** \( \mathbf{V}(r, t) \) and \( \mathbf{v}(t) !!

\[
\mathbf{J}_\alpha(r, t) = q_\alpha \mathbf{V}_\alpha = q_\alpha n_\alpha \mathbf{V}_\alpha \quad \text{Electric current density, } \{J\} = \text{C} \text{m}^{-2} \text{s}^{-1} = \text{A} \text{m}^{-2}
\]

Pressure and temperature

from the second velocity moments

Pressure tensor

\[
\mathbf{P}_\alpha(r, t) = m_\alpha \int (v - \mathbf{V}_\alpha)(v - \mathbf{V}_\alpha) f_\alpha(r, v, t) \, d^3v
\]

\( \text{dyadic product } \rightarrow \text{tensor} \)

If \( \mathbf{P}_\alpha = p_\alpha \mathbf{I} \) where \( \mathbf{I} \) is the unit tensor, we find the scalar pressure

\[
p_\alpha = \frac{m_\alpha}{3} \int (v - \mathbf{V}_\alpha)^2 f_\alpha(r, v, t) \, d^3v = n_\alpha k_B T_\alpha \quad \text{introducing the temperature}
\]

Assume \( \mathbf{V} = 0 \):

\[
\frac{3}{2} k_B T_\alpha(r, t) = \frac{m_\alpha}{2} \frac{\int f_\alpha(r, v, t) \, d^3v}{\int f_\alpha(r, v, t) \, d^3v} \quad T \propto \langle \mathbf{K.E.} \rangle
\]

Thus we can calculate a "temperature" also in non-Maxwellian plasma!

Magnetic pressure

(i.e. magnetic energy density)

\[
\frac{B^2}{2\mu_0}
\]

Plasma beta

\[
\beta = \frac{2\mu_0}{B^2} \sum \alpha n_\alpha k_B T_\alpha \quad \beta \ll 1 \quad \text{B dominates over plasma}
\]

\[
\beta \gg 1 \quad \text{plasma dominates over B}
\]

thermal pressure / magnetic pressure

3rd velocity moment \( \rightarrow \text{heat flux} \) (temperature x velocity), etc. to higher orders…
Macroscopic plasma description

Macroscopic plasma theories are fluid theories at different levels

- single fluid (magnetohydrodynamics MHD)
- two-fluid (multifluid, separate equations for electron and ion fluids)
- hybrid (fluid electrons with (quasi)particle ions)

Macroscopic equations can be obtained by taking velocity moments of Boltzmann / Vlasov equations

\[
\int v^n \left( \frac{\partial f_n}{\partial t} + v \cdot \frac{\partial f_n}{\partial x} + \frac{q_n}{m_n} (E + v \times B) \cdot \frac{\partial f_n}{\partial v} \right) d^3v = \int \left( \frac{\partial f_n}{\partial t} \right)_c d^3v
\]

Taking the \(n^{th}\) moment of BE/VE introduces terms of order \(n + 1\)!
This leads to an open chain of equations that must be terminated by applying some form of physical intuition.
Note that the collision integrals can be very tricky!

We start from the Boltzmann equation

\[
\frac{\partial f_n}{\partial t} + v \cdot \frac{\partial f_n}{\partial x} + \frac{q_n}{m_n} (E + v \times B) \cdot \frac{\partial f_n}{\partial v} = \left( \frac{\partial f_n}{\partial t} \right)_c
\]

and calculate its zeroth velocity moment.

In absence of ionizing or recombining collisions, the collision integral is zero, and the result is the continuity equation

\[
\frac{\partial n_{\text{tot}}}{\partial t} + \nabla \cdot (n_{\text{tot}} V_{\text{tot}}) = 0
\]

General form of conservation law for \(F\):

\[
\frac{\partial F}{\partial t} + \nabla \cdot G = 0
\]

To calculate the first moment, multiply BE by momentum \(m_n v\) and integrate

\[
\Rightarrow \quad n_{\text{tot}} m_n \frac{\partial V_{\text{tot}}}{\partial t} + n_{\text{tot}} m_n V_{\text{tot}} \cdot \nabla V_{\text{tot}} - n_{\text{tot}} q_{\text{tot}} (E + V_{\text{tot}} \times B) + \nabla \cdot P_{\text{tot}} = m_n \int v \left( \frac{\partial f_n}{\partial t} \right)_c d^3v.
\]

Equation for momentum transport, actually equation of motion!
Now the convective derivative of \( \nabla \cdot \mathbf{V}_\alpha \) and the pressure tensor \( \mathcal{P}_\alpha \) are second moments.

The electric and magnetic fields must fulfill Maxwell’s equations:

\[
\nabla \cdot \mathbf{E} = \sum \frac{n_{\alpha} q_{\alpha}}{\epsilon_0} + \frac{\rho_{ext}}{\epsilon_0} \\
\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \sum n_{\alpha} q_{\alpha} \mathbf{V}_\alpha + \mu_0 \mathbf{J}_{ext}
\]

\((\rho_{ext}, \mathbf{J}_{ext})\) are external sources.

Note that the collision integral can be non-zero, because collisions transfer momentum between different particle species!

Calculate the second moment (multiply by \( \mathbf{V}_\alpha \), and integrate; rather tedious) — heat transfer equation (conservation of energy)

Now the heat flux is of third order. To close the chain some equation relating the variables must be introduced.

---

**Equations of MHD**

Sum over all particle species:

\[
\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0
\]

\[
\rho_m \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} + \nabla \mathbf{p} - \mathbf{J} \times \mathbf{B} = 0 \quad \text{(isotropic pressure assumed)}
\]

\[
\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{J}/\sigma \quad \text{(or energy equation)}
\]

Relevant Maxwell’s equations; displacement current neglected:

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \Rightarrow \nabla \cdot \mathbf{J} = 0
\]

In space plasmas the conductivity often is very large: ideal MHD \( \mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \)

However, sometimes other terms than the resistive start to play a role (e.g., in magnetic reconnection) and a more general Ohm’s law is needed:

\[
\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{\mathbf{J}}{\sigma} + \frac{1}{ne} \mathbf{J} \times \mathbf{B} - \frac{1}{ne} \nabla \cdot \mathcal{P}_e + \frac{m_e}{ne^2} \frac{\partial \mathbf{J}}{\partial t}
\]
Convection and diffusion of $B$

Take curl of the MHD Ohm’s law $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{J}/\sigma$ and apply Faraday’s law

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{V} \times \mathbf{B} - \mathbf{J}/\sigma)$$

Thereafter use Ampère’s law and the divergence of $\mathbf{B}$ to get the induction equation for the magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

(Note that $\sigma$ has been assumed constant here)

Assume that plasma does not move ($\mathbf{V} = 0$)

$$\Rightarrow \text{diffusion equation: } \frac{\partial \mathbf{B}}{\partial t} = D_m \nabla^2 \mathbf{B} \quad \text{diffusion coefficient: } D_m = (\mu_0 \sigma)^{-1}$$

If the resistivity is finite, the magnetic field diffuses into the plasma to remove local magnetic inhomogeneities, e.g., curves in the field, etc.

Let $L_B$ the characteristic scale of magnetic inhomogeneities. The solution is

$$\mathbf{B} = B_0 \exp(\pm t/\tau_d) \quad \text{where the characteristic diffusion time is } \tau_d = \mu_0 \sigma L_B^2$$

In case of $\sigma \to \infty$ the diffusion becomes very slow and the evolution of $\mathbf{B}$ is completely determined by the plasma flow (field is frozen-in to the plasma)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \quad \text{convection equation}$$

Let the characteristic spatial and temporal scales be

$$\nabla \to L_B^{-1} \quad \& \quad \partial / \partial t \to \tau^{-1} \quad \text{and the diffusion time } \tau_d = \mu_0 \sigma L_B^2$$

The order of magnitude estimates for the terms of the induction equation are

$$\frac{\mathbf{B}}{\tau} = \frac{V \mathbf{B}}{L_B} + \frac{\mathbf{B}}{\tau_d}$$

The measure of the relative strengths of convection and diffusion is the magnetic Reynolds number $R_m = \mu_0 \sigma L_B V = L_B V / D_m$.

This is analogous to the Reynolds number in hydrodynamics $R = LV/\nu$.

In fully ionized plasmas $R_m$ is often very large. E.g. in the solar wind at 1 AU it is $10^{16} - 10^{17}$. This means that during the 150 million km travel from the Sun the field diffuses about 1 km! Very ideal MHD:

$$\mathbf{E} = - \mathbf{V} \times \mathbf{B}$$
Break-down of the frozen-in condition: Magnetic reconnection

Magnetic reconnection is a fundamental energy release process in magnetized plasmas but we skip the details on this lecture.

Magnetohydrodynamic waves
Alfvén waves

MHD is a fluid theory and there are similar wave modes as in ordinary fluid theory (hydrodynamics). In hydrodynamics the restoring forces for perturbations are the pressure gradient and gravity. Also in MHD the pressure force leads to acoustic fluctuations, whereas Ampère’s force (JxB) leads to an entirely new class of wave modes, called Alfvén (or MHD) waves.

As the displacement current $\varepsilon^{-2} \partial E/\partial t$ is neglected in MHD, there are no electromagnetic waves of classical electrodynamics. Of course EM waves can propagate through MHD plasma (e.g. light, radio waves, etc.) and even interact with the plasma particles, but that is beyond the MHD approximation.
Dispersion equation for ideal MHD waves

\[
\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0
\]

\[
\rho_m \frac{\partial \mathbf{V}}{\partial t} + \rho_m (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla p + \mathbf{J} \times \mathbf{B}
\]

\[
\nabla p = \frac{\partial \rho_m}{\partial t}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}
\]

\[
\nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}
\]

\[
\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0
\]

Eliminate \( \mathbf{J} \), \( p \), and \( \mathbf{E} \)

\[
\rho_m \frac{\partial \mathbf{V}}{\partial t} + \rho_m (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla \rho_m + (\nabla \times \mathbf{B})/\mu_0
\]

\[
\nabla \times (\mathbf{V} \times \mathbf{B}) = \frac{\partial \mathbf{B}}{\partial t}
\]

We are left with 7 scalar equations for 7 unknowns \((\rho_{m0}, \mathbf{V}, \mathbf{B})\)

Consider small perturbations

\[
\mathbf{B}(r, t) = B_0 + B_1(r, t)
\]

\[
\rho_m(r, t) = \rho_{m0} + \rho_{m1}(r, t)
\]

\[
\mathbf{V}(r, t) = \mathbf{V}_1(r, t)
\]

and linearize

Find an equation for \( \mathbf{V}_1 \). Start by taking the time derivative of \( \mathbf{B}_1 \)

\[
\rho_{m0} \frac{\partial \mathbf{V}_1}{\partial t} + \rho_{m1} (\nabla \cdot \mathbf{V}_1) = 0 \quad (\ast)
\]

\[
\rho_{m0} \frac{\partial \mathbf{V}_1}{\partial t} + \rho_{m1} (\mathbf{V} \cdot \nabla) \mathbf{V}_1 + \mathbf{B}_0 \times (\nabla \times \mathbf{B}_1)/\mu_0 = 0 \quad (\ast\ast)
\]

\[
\frac{\partial \mathbf{B}_1}{\partial t} - \nabla \times (\mathbf{V}_1 \times \mathbf{B}_0) = 0 \quad (\ast\ast\ast)
\]

Insert \( \ast \) and \( \ast\ast\ast \) and introduce the Alfvén velocity as a vector \( \mathbf{V}_A = \frac{B_0}{\sqrt{\mu_0 \rho_{m0}}} \)

Find an equation for \( \mathbf{V}_1 \). Start by taking the time derivative of \( \ast \ast \ast \)

\[
\rho_{m0} \frac{\partial^2 \mathbf{V}_1}{\partial t^2} + \rho_{m1} \nabla (\nabla \cdot \mathbf{V}_1) + \mathbf{B}_0 \times \left( \nabla \times \frac{\partial \mathbf{B}_1}{\partial t} \right) = 0
\]

Look for plane wave solutions \( \mathbf{V}_1(r, t) = \mathbf{V}_1 \exp[i(k \cdot r - \omega t)] \)

\[
-\omega^2 \mathbf{V}_1 + c_A^2 (\mathbf{k} \cdot \mathbf{V}_1) \mathbf{k} - \mathbf{V}_A \times \left( \mathbf{k} \times (\mathbf{V}_1 \times \mathbf{V}_A) \right) = 0
\]

Using \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C} \) a few times we have the dispersion equation for the waves in ideal MHD

\[
-\omega^2 \mathbf{V}_1 + (c_A^2 + v_A^2)(\mathbf{k} \cdot \mathbf{V}_1) \mathbf{k} + (\mathbf{k} \cdot \mathbf{V}_A)[(\mathbf{k} \cdot \mathbf{V}_A) \mathbf{V}_1 - (\mathbf{V}_A \cdot \mathbf{V}_1) \mathbf{k} - (\mathbf{k} \cdot \mathbf{V}_1)\mathbf{V}_A] = 0
\]
Propagation perpendicular to the magnetic field: \( k \perp B_0 \)

Now \( k \cdot v_A = 0 \) and the dispersion equation reduces to

\[
-\omega^2 V_1 + (c_s^2 + v_A^2)(k \cdot V_1)k = 0
\]

\[
\Rightarrow \quad V_1 = (c_s^2 + v_A^2)(k \cdot V_1)k/\omega^2 \quad \text{clearly} \quad k \parallel V_1
\]

And we have found the magnetosonic wave

\[ \omega/k = \sqrt{c_s^2 + v_A^2} \]

This mode has many names in the literature: compressional Alfvén wave, fast Alfvén wave, fast MHD wave

Propagation parallel to the magnetic field: \( k \parallel B_0 \)

\[
(k^2 v_A^2 - \omega^2)V_1 + \left( \frac{c_s^2}{v_A^2} - 1 \right) k^2 (V_1 \cdot v_A) v_A = 0
\]

Two different solutions (modes)

1) \( V_1 \parallel B_0 \parallel k \Rightarrow \quad \omega/k = c_s \quad \text{the sound wave} \)

2) \( V_1 \perp B_0 \parallel k \Rightarrow \quad V_1 \cdot v_A = 0 \)

\[
\Rightarrow \quad \omega/k = v_A
\]

This mode is called Alfvén wave or shear Alfvén wave

Propagation at an arbitrary angle

\[
\begin{align*}
k &= k(e_x \sin \theta + e_z \cos \theta) \\
v_A &= v_A e_z \\
V_1 &= V_{1x} e_x + V_{1y} e_y + V_{1z} e_z \\
k \cdot v_A &= k v_A \cos \theta \\
k \cdot V_1 &= k(V_{1x} \sin \theta + V_{1z} \cos \theta) \\
v_A \cdot V_1 &= v_A V_{1z} \\
\end{align*}
\]

Dispersion equation \( \Rightarrow \)

\[
\begin{align*}
V_{1x}(-\omega^2 + k^2 v_A^2 + k^2 v_A^2 \sin^2 \theta) + V_{1z}(k^2 v_A^2 \sin \theta \cos \theta) &= 0 \\
V_{1y}(-\omega^2 + k^2 v_A^2 \cos^2 \theta) &= 0 \\
V_{1x}(k^2 v_A^2 \sin \theta \cos \theta) + V_{1z}(-\omega^2 + k^2 v_A^2 \cos^2 \theta) &= 0
\end{align*}
\]

Coeff. of \( V_{1y} \Rightarrow \quad \omega/k = v_A \cos \theta \quad \text{shear Alfvén wave} \)

From the determinant of the remaining equations:

\[
\left( \frac{\omega}{k} \right)^2 = \frac{1}{2}(c_s^2 + v_A^2) \pm \frac{1}{2} \left[ (c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2 \theta \right]^{1/2}
\]

Fast (+) and slow (−) Alfvén/MHD waves
Some remarks

- Collective effects of free charges determine the behavior of the plasma as an electromagnetic medium
- Plasma physics also relies on tools of statistical physics
- Plasma behaves nonlinearly
  - Vlasov equation is nonlinear, magnetohydrostatic equilibrium is nonlinear, etc.
  - Linearizations are often useful, e.g., to find the normal modes of plasma oscillations, but the observable plasma oscillations are either damped or grow to nonlinear level leading to instabilities
- Plasma is often turbulent
  - Plasma turbulence is an even more complicated issue than ordinary fluid turbulence
- Plasma systems often exhibit chaotic behavior
  - Concepts of chaos, such as self-organized criticality, intermittence, renormalization groups, etc., are important in theoretical plasma physics.